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Lipschitz inverse and direct sequences

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Abstract

We extend some results of the author, concerning inverse sequences with bonding maps being weak contractions, over the class of Lipschitz inverse sequences. Further, we define and study Lipschitz direct sequences. Finally, we examine some sequences of sets with positive reach in $\mathbb{R}^k$ and apply Lipschitz inverse and direct sequences to prove the continuity of Hausdorff measure for manifolds with positive reach.

Key words: Lipschitz inverse sequences; Lipschitz direct sequences; Metric limit; Hausdorff limit; Hausdorff measure; Compact manifold; positive reach

AMS (MOS) Sub. Class.: 54B35, 28A78, 57N99, 54E45

Introduction

The notion of metric inverse limit was introduced in [5] and studied by Rudnik in [6].

Let $A=(A_n, p_n^{n+1})$ be an inverse sequence with all A_n being nonempty subsets of a metric space (X, ρ), with all p_n^{n+1} continuous and all the threads convergent in (X, ρ). Let $\varprojlim A$ be the topological inverse limit of A and let

$$\underset{\twoheadleftarrow}{\lim} A := \left\{\lim_n x_n \colon (x_n) \in \varprojlim A\right\}.$$

If the quotient map $h \colon \varprojlim A \to \underset{\twoheadleftarrow}{\lim} A$ defined by

$$h(\boldsymbol{x}) := \lim_n x_n \quad \text{for every thread } \boldsymbol{x}=(x_n)$$

is a homeomorphism, then $\underset{\twoheadleftarrow}{\lim} A$ is an inverse limit of A in the category $\mathbf{Top}_0$ (whose objects are metric spaces), and the formula

$$p_m\left(\lim_n x_n\right) = x_m \quad \text{for every thread } (x_n)$$

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defines the sequence of projections $p_m : \varprojlim A \to A_m$ (see [5, Proposition 1.4]). In this situation we refer to $\varprojlim A$ as the *metric inverse limit of* A.

If X and all A_n are compact, all p_n^{n+1} surjective, and all the threads are equiconvergent, then $\varprojlim A$ coincides with the Hausdorff limit of (A_n) (see [6, Theorem 0.3.5]). According to [5], an inverse sequence A is called *geometric* if h is a homeomorphism and all the threads of A are equiconvergent.

In particular, for the category **Metr**, whose morphisms are weak contractions, geometric inverse sequences of closed subsets of a compact space (X, ρ) are characterized by the equiconvergence of the threads [5, 4.5]. If, in addition, all the sets are connected and the bonding maps surjective, then the 1-dimensional Hausdorff measure $\mathscr{H}^1$ is continuous with respect to $\varprojlim$ [5, Theorem 4.6].

In Sections 1 and 2 of the present paper the results of [5, Section 4] are generalized to the category **Lip**, whose morphisms are Lipschitz maps. In particular, it turns out that for the so-called Lipschitz inverse sequences (Definition 1.1) the characterization of geometric inverse sequences is exactly the same as in [5, 4.5] for **Metr** (see Theorem 2.2 below). It also turns out that Theorem 4.6 of [5] concerning the continuity of $\mathscr{H}^1$ holds for arbitrary Lipschitz inverse sequences, not only for those in **Metr** (see Theorem 2.3 below, which contains also an inequality for $\mathscr{H}^s$ for arbitrary $s > 0$).

Further, we are concerned with the continuity of Hausdorff measure $\mathscr{H}^s$ with respect to $\lim_{\mathrm{H}}$. Corollary 4.9 of [5] deals with a very special case: A_n are the boundaries of convex bodies in $\mathbb{R}^2$. In Section 5 we consider sequences of compact sets with positive reach in $\mathbb{R}^k$ and, in particular, sequences of compact r-manifolds with positive reach for $1 \leqslant r \leqslant k-1$. Using Lipschitz inverse sequences we can prove only one inequality for the Hausdorff measure $\mathscr{H}^r$. To obtain the other one, we use Lipschitz direct sequences, which are introduced and discussed in Sections 3 and 4. This leads to Corollary 5.7.

As was recently noticed by S. Spież, Corollary 5.7 can be derived from Theorem 5.9 and Remark 6.14 of Federer's paper [2]. However Federer's proof requires the whole complicated machinery of curvature measures, while our method seems transparent and simple.

In what follows, the *Lipschitz constant of a Lipschitz map* $f : X \to Y$ is understood as

$$\mathrm{Lip}(f) := \inf\{\lambda > 0\colon (\forall x,\, y \in X)[\rho(f(x),\, f(y)) \leqslant \lambda\rho(x,\, y)]\}.$$

In contradistinction to the common notation, for any sequence (λ_n) of nonnegative numbers we use the symbol $\prod_{n=1}^{\infty}\lambda_n$ to denote $\lim_i \prod_{n=1}^{i}\lambda_n$, even if this limit is 0 or ∞; thus we write

$$0 < \prod_{n=1}^{\infty}\lambda_n \quad \text{or} \quad \prod_{n=1}^{\infty}\lambda_n < \infty$$

if, in particular, it is positive or finite.

As above, we use the notation (x_n) instead of $(x_n)_{n \in \mathbb{N}}$ for a sequence when there is no ambiguity.

By a map we always mean a continuous function.

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1. Lipschitz inverse sequences

Let (X, ρ) be a metric space. We are interested in some special inverse sequences in (X, ρ).

Definition 1.1. An inverse sequence $A = (A_n, p_n^{n+1})$ in (X, ρ) is *Lipschitzian* if and only if all p_n^{n+1} are Lipschitz maps and

$$\liminf_k \prod_{n=1}^{k} \operatorname{Lip}(p_n^{n+1}) < \infty.$$

Let $\operatorname{Lip}(A) := \liminf_k \prod_{n=1}^{k} \operatorname{Lip}(p_n^{n+1})$. We shall refer to $\operatorname{Lip}(A)$ as the *Lipschitz constant of* A.

For instance, every inverse sequence $A = (A_n, p_n^{n+1})$ in the category **Metr** is Lipschitzian. Indeed, if the bonding maps are weak contractions, then $\operatorname{Lip}(p_n^{n+1}) \leqslant 1$ for every n and thus

$$0 \leqslant \prod_{n=1}^{\infty} \operatorname{Lip}(p_n^{n+1}) < \infty.$$

We shall first examine Lipschitz inverse sequences with Lipschitz constant equal to 0 (Proposition 1.3). Let us begin with the following.

Lemma 1.2. *Let $A = (A_n, p_n^{n+1})$ be a Lipschitz inverse sequence in (X, ρ). Let $\lambda = \operatorname{Lip}(A)$ and $\lambda_n = \operatorname{Lip}(p_n^{n+1}) > 0$ for every n. Then, for every two convergent threads $\boldsymbol{x} = (x_n)$ and $\boldsymbol{y} = (y_n)$ with $\lim_n x_n = x$ and $\lim_n y_n = y$,*

$$\rho(x_1, y_1) \leqslant \lambda \rho(x, y)$$

and

$$\rho(x_i, y_i) \leqslant \left(\prod_{n=1}^{i-1} \lambda_n \right)^{-1} \lambda \rho(x, y) \text{ for every } i \geqslant 2.$$

Proof. For every $k \geqslant 2$,

$$\rho(x_1, y_1) \leqslant \lambda_1 \rho(x_2, y_2) \leqslant \cdots \leqslant \prod_{n=1}^{k-1} \lambda_n \rho(x_k, y_k).$$

Since $\lim_k \rho(x_k, y_k) = \rho(x, y)$, it follows that

$$\liminf_k \left(\prod_{n=1}^{k-1} \lambda_n \rho(x_k, y_k) \right) = \lambda \rho(x, y),$$

which implies both inequalities. □

Proposition 1.3. *Let A be a Lipschitz inverse sequence in (X, ρ) with all the threads convergent. If* $\operatorname{Lip}(A) = 0$, *then either $\underleftarrow{\lim} A$ is a singleton or there exists n_0 such that* $\operatorname{Lip}(A \mid n \geqslant n_0) > 0$.

Proof. Let $\lambda_n = \operatorname{Lip}(p_n^{n+1})$ for every n. Since $\operatorname{Lip}(A) = 0$, from Lemma 1.2 it follows that if $\lambda_n > 0$ for every n, then $\underleftarrow{\lim} A$ is a singleton. Thus it suffices to consider the following two cases:

Case 1: $\lambda_{k_n} = 0$ *for some increasing sequence* (k_n). Then, evidently, A has a unique thread, that is, $\underleftarrow{\lim} A$ is a singleton.

Case 2: *There exists n_0 such that $\lambda_n > 0$ for every $n \geqslant n_0$*. Then either $\operatorname{Lip}(A \mid n \geqslant n_0) > 0$ or $\operatorname{Lip}(A \mid n \geqslant n_0) = 0$. In the latter case, applying Lemma 1.2 to $A \mid n \geqslant n_0$, we infer that $\underleftarrow{\lim} A$ is a singleton (because $\underleftarrow{\lim} A$ is homeomorphic to $\underleftarrow{\lim}(A \mid n \geqslant n_0)$). □

We shall need the following.

Lemma 1.4 (compare [5, 4.1]). *Let $A = (A_n, p_n^{n+1})$ be a Lipschitz inverse sequence in (X, ρ) with* $\operatorname{Lip}(p_n^{n+1}) > 0$ *for every n. If all the threads of A are convergent in (X, ρ), then*

(i) *the quotient map h is injective and h^{-1} continuous,*

(ii) *each p_m is Lipschitzian,*

(iii) $\limsup_m \operatorname{Lip}(p_m) \leqslant 1$.

Proof. Let $\lambda_n = \operatorname{Lip}(p_n^{n+1})$ for every n and $\lambda = \operatorname{Lip}(A)$. By Lemma 1.2, for any threads $\boldsymbol{x} = (x_n)$ and $\boldsymbol{y} = (y_n)$ with $\lim_n x_n = x$ and $\lim_n y_n = y$,

$$\rho(x_1, y_1) \leqslant \lambda \rho(x, y)$$

and (1.1)

$$\rho(x_m, y_m) \leqslant \lambda \left(\prod_{n=1}^{m-1} \lambda_n \right)^{-1} \rho(x, y) \quad \text{for } m \geqslant 2.$$

Thus h is injective and, since $x_m = p_m(x)$ for every m, each p_m is Lipschitzian. This proves (ii). Hence h^{-1} is continuous, which proves (i). By condition (1.1),

$$\operatorname{Lip}(p_1) \leqslant \lambda \quad \text{and} \quad \operatorname{Lip}(p_m) \leqslant \lambda \left(\prod_{n=1}^{m-1} \lambda_n \right)^{-1} \quad \text{for m} \geqslant 2.$$

This implies (iii). □

2. Geometric Lipschitz inverse sequences

To characterize geometric Lipschitz inverse sequences we need the following lemma (see [3, 2.10.21]).

Lemma 2.1. *If (X, ρ) is compact and a sequence of Lipschitz maps $f_n : X \to Y$ with $\sup_n \mathrm{Lip}(f_n) < \infty$ is pointwise convergent to a Lipschitz map f, then (f_n) is uniformly convergent to f.*

Let us prove the following.

Theorem 2.2 (compare [5, 4.5]). *Let $A = (A_n, p_n^{n+1})$ be a Lipschitz inverse sequence in a compact space (X, ρ), with all A_n compact. Then the following are equivalent:*

(i) *A is a geometric inverse sequence,*
(ii) *the threads of A are equiconvergent,*
(iii) *the threads are convergent and $\varprojlim A$ is compact.*

Proof. In view of Proposition 1.3, we may assume that $\mathrm{Lip}(A) > 0$. Let $\alpha_n = \mathrm{Lip}(p_n)$ for every n. By Lemma 1.4(iii), $\limsup_n \alpha_n \leqslant 1$. Thus the argument is the same as in the proof of Theorem 4.5 of [5], with Lemma 4.1 of [5] replaced by Lemma 1.4 (i), (ii) and Lemma 4.4 of [5] replaced by Lemma 2.1. □

We complete this section with the following.

Theorem 2.3. *Let $A = (A_n, p_n^{n+1})$ be a Lipschitz inverse sequence in a compact space (X, ρ), with all A_n being continua, all p_n^{n+1} surjective, and all the threads convergent. Let $s > 0$ and $H^s(A_n) < \infty$ for every n. Then*

(i) $\limsup_n \mathscr{H}^s(A_n) \leqslant \mathscr{H}^s(\varprojlim A)$,
(ii) *if* $\mathrm{Lip}(A) = \prod_{n=1}^{\infty} \mathrm{Lip}(p_n^{n+1})$, *then*
  (a) *the sequence $(\mathscr{H}^s(A_n))$ has a (finite or infinite) limit,*
  (b) *if, in addition, the threads of A are equiconvergent and $s = 1$, then* $\lim_n \mathscr{H}^1(A_n) = \mathscr{H}^1(\varprojlim A)$.

Proof. In view of Proposition 1.3, we may assume that $\mathrm{Lip}(A) > 0$. Let

$$\lambda_n = \mathrm{Lip}(p_n^{n+1}), \qquad \lambda = \mathrm{Lip}(A), \quad \text{and} \quad A = \varprojlim A.$$

By Lemma 1.4(ii), each p_n is Lipschitzian. Let $\alpha_n = \mathrm{Lip}(p_n)$ for every n. Then, by [1, 1.8],

$$\mathscr{H}^s(A_n) \leqslant (\alpha_n)^s \mathscr{H}^s(A) \quad \text{for every } n,$$

which, together with Lemma 1.4(iii), proves (i).

Let now $\lambda = \prod_{n=1}^{\infty} \lambda_n$. Applying again [1, 1.8], we obtain

$$\mathscr{H}^s(A_1) \leqslant (\lambda_1)^s \mathscr{H}^s(A_2) \leqslant \cdots \leqslant \left(\prod_{i=1}^{n-1} \lambda_i\right)^s \mathscr{H}^s(A_n) \quad \text{for } n \geqslant 2,$$

whence the sequence (β_n) defined by

$$\beta_1 = 1, \quad \beta_n = \left(\prod_{i=1}^{n-1} \lambda_i\right)^s \mathscr{H}^s(A_n) \quad \text{for } n \geqslant 2$$

is either convergent or divergent to ∞. Since

$$\lim_n \left(\prod_{i=1}^{n-1} \lambda_i\right)^s = \lambda^s,$$

the sequence $(\mathscr{H}^s(A_n))_{n \in \mathbb{N}}$ is either convergent or divergent to ∞, which proves (iia).

If the threads are equiconvergent, then, by [6, 0.3.5], $\underleftarrow{\lim}\, A = \lim_{\mathrm{H}} A_n$ and thus we may apply Gołąb's theorem [1, 3.18] to obtain

$$\mathscr{H}^1(A) \leqslant \lim_n \mathscr{H}^1(A_n).$$

This inequality, together with (i), proves (iib). □

The following example shows that in Theorem 2.3 the assumption on A to be Lipschitzian is essential.

Example 2.4. Let $\varphi_i : \mathbb{R}^2 \to \mathbb{R}^2$ be the homothety with centre $((-1)^i, 0)$ and ratio $\frac{1}{2}$ for $i = 1, 2$. We define $A = (A_n, p_n^{n+1})$ as follows:

$$A_1 = \{(t, 1 - |t|): t \in [-1, 1]\}, \qquad A_n = \varphi_1(A_{n-1}) \cup \varphi_2(A_{n-1}) \quad \text{for } n \geqslant 2,$$

and p_n^{n+1} is the projection parallel to the vector $e_2 = (0, 1)$. Then A is a geometric inverse sequence with $\underleftarrow{\lim}\, A = [-1, 1] \times \{0\}$. Each p_n^{n+1} is Lipschitzian with $\mathrm{Lip}(p_n^{n+1}) = \sqrt{2}$, whence A is not Lipschitzian. It is easy to see that $\mathscr{H}^1(A_n) = 2\sqrt{2}$ for every n and $\mathscr{H}^1(\underleftarrow{\lim}\, A) = 2$, whence

$$\lim_n \mathscr{H}^1(A_n) \neq \mathscr{H}^1(\underleftarrow{\lim}\, A).$$

3. Metric limits of direct sequences

We shall restrict our consideration to direct sequences in a metric space (X, ρ), with all the bonding maps surjective.

Definition 3.1. Let $A = (A_n, p_{n+1}^n)$ be a direct sequence in (X, ρ) (i.e., all A_n are nonempty subsets of X), and let $p_{n+1}^n : A_n \to A_{n+1}$ be surjective for every n. A sequence $(x_n) \in \mathrm{P}_{n=1}^{\infty} A_n$ is a *thread of* A if and only if

$$p_{n+1}^n(x_n) = x_{n+1} \quad \text{for every } n.$$

Let $\mathrm{Thr}(A)$ be the set of all threads of A, with topology of subspace of $\mathrm{P}_{n=1}^{\infty} A_n$.

Definition 3.2. Let A be a direct sequence in (X, ρ), with the bonding maps surjective and all the threads convergent in (X, ρ). The set $\varinjlim A$ defined by

$$\varinjlim A = \left\{\lim_n (x_n) : (x_n) \in \mathrm{Thr}(A)\right\}$$

will be called the *metric limit of* A.

We use the common notation:

$$p_n^n = \mathrm{id}_{A_n} \quad \text{and} \quad p_n^m = p_n^{n-1} \cdots p_{m+1}^m \quad \text{for } m < n-1.$$

We need the following.

Lemma 3.3. *If a direct sequence A in (X, ρ) with surjective bonding maps has the threads equiconvergent, then for every convergent sequence $(x^{(k)})_{k \in \mathbb{N}}$ of threads of A, if $x^{(k)} = (x_n^{(k)})_{n \in \mathbb{N}}$, then*

$$\lim_k \lim_n x_n^{(k)} = \lim_n \lim_k x_n^{(k)}.$$

Proof. Let

$$x^{(k)} = \lim_n x_n^{(k)} \quad \text{for every } k$$

and (3.1)

$$x_n = \lim_k x_n^{(k)} \quad \text{for every } n.$$

Then $(x_n)_{n \in \mathbb{N}} \in \mathrm{Thr}(A)$, because continuity of the bonding maps implies that $\mathrm{Thr}(A)$ is closed in $\mathrm{P}_{n=1}^{\infty} A_n$. Let $x = \lim_n x_n$. We have to prove that

$$x = \lim_k x^{(k)}. \tag{3.2}$$

Let $\epsilon > 0$. By the equiconvergence of all $x^{(k)}$ and $x = (x_n)$, there exists n_0 such that

$$(\forall n \geqslant n_0)(\forall k)\left[\rho\left(x_n^{(k)}, x^{(k)}\right) < \frac{\epsilon}{3} \text{ and } \rho(x_n, x) < \frac{\epsilon}{3}\right].$$

By the second part of (3.1), there exists $(k_n)_{n \in \mathbb{N}}$ such that

$$(\forall k \geqslant k_n)\left[\rho\left(x_n^{(k)}, x_n\right) < \frac{\epsilon}{3}\right].$$

Let $k_0 := k_{n_0}$. Then for every $k \geqslant k_0$,

$$\rho(x^{(k)}, x) \leqslant \rho\left(x^{(k)}, x_{n_0}^{(k)}\right) + \rho\left(x_{n_0}^{(k)}, x_{n_0}\right) + \rho(x_{n_0}, x) < \epsilon,$$

which proves (3.2). □

Proposition 3.4. *Let $A = (A_n, p_{n+1}^n)$ be a direct sequence in (X, ρ) with surjective bonding maps and all the threads equiconvergent. Then the formula*

$$p^m(x) = \lim_{n \geqslant m} p_n^m(x) \quad \textit{for } m \in \mathbb{N} \tag{3.3}$$

defines a sequence of maps $p^m : A_m \to \varinjlim A$ *such that*

$$p^{m+1} \circ p^m_{m+1} = p^m \quad \textit{for every } m.$$

Proof. For every $x \in A_m$ the sequence $(p^m_n(x))_{n \geqslant m}$ can be extended to a thread $(x_n)_{n \in \mathbb{N}} \in \mathrm{Thr}(A)$. If (x_n) and (y_n) are such extensions for the same x, then, evidently, $\lim_n x_n = \lim_n y_n$. Thus for every m the formula (3.3) defined a function $p^m : A_m \to \varinjlim A$.

The function p^m is continuous; indeed, for any sequence $(x^{(k)})_{k \in \mathbb{N}}$ in A_m, if

$$x = \lim_k x^{(k)} \in A_m,$$

then

$$p^m(x^{(k)}) = \lim_{n \geqslant m} p^m_n(x^{(k)}),$$

whence, by Lemma 3.3 and by the continuity of p^m_n,

$$\begin{aligned}\lim_k p^m(x^{(k)}) &= \lim_k \lim_{n \geqslant m} p^m_n(x^{(k)}) \\ &= \lim_{n \geqslant m} \lim_k p^m_n(x^{(k)}) = \lim_{n \geqslant m} p^m_n(x) = p^m(x).\end{aligned}$$

Finally, the sequence (p^m) satisfies the required equality. □

The maps $p^m : A_m \to \varinjlim A$ will be called the *projections*.

We shall now prove that the relationship between the Hausdorff and the metric limit for direct sequences is analogous to that for inverse sequences (compare [6, 0.3.5]).

Theorem 3.5. *Let* $A = (A_n, p^n_{n+1})$ *be a direct sequence in* (X, ρ), *with surjective bonding maps, all the threads equiconvergent, and all* A_n *compact. Then*

$$\varinjlim A = \lim{}_{\mathrm{H}} A_n.$$

Proof. Let $A = \varinjlim A$. Since $A = p^n(A_n)$ (for arbitrary n) and p^n is continuous, it follows that A is compact.

Take $\epsilon > 0$. On the one hand, for every $a \in A$ there exists $(a_n) \in \mathrm{Thr}(A)$ with $a = \lim_n a_n$. Since all the threads are equiconvergent, there exists n_0 (independent of a) such that

$$(\forall n \geqslant n_0)[\rho(a, A_n) \leqslant \rho(a, a_n) < \epsilon];$$

hence

$$A \subset (A_n)_\epsilon \quad \text{for } n \geqslant n_0. \tag{3.4}$$

On the other hand, if $x \in A_m$ for some $m \geqslant n_0$, then there exists $(a_n) \in \mathrm{Thr}(A)$ with $a_m = x$, whence

$$\rho(x, A) \leqslant \rho\left(a_m, \lim_n a_n\right) < \epsilon.$$

Thus

$$A_m \subset (A)_\epsilon \quad \text{for } m \geqslant n_0. \tag{3.5}$$

From (3.4) and (3.5) it follows that for $n \geqslant n_0$

$$\rho_H(A, A_n) < \epsilon.$$

This completes the proof. □

4. Lipschitz direct sequences

Definition 4.1. A direct sequence $A = (A_n, p_{n+1}^n)$ in (X, ρ) is *Lipschitzian* if and only if all p_{n+1}^n are surjective Lipschitz maps and

$$0 \leqslant \liminf_k \prod_{n=1}^{k} \mathrm{Lip}(p_{n+1}^n) < \infty.$$

We shall refer to $\liminf_k \prod_{n=1}^k \mathrm{Lip}(p_{n+1}^n)$ as *Lipschitz constant of* A, in symbols $\mathrm{Lip}(A)$.

Let us prove the following

Proposition 4.2. *Let $A = (A_n, p_{n+1}^n)$ be a Lipschitz direct sequence in (X, ρ), with all the threads equiconvergent. Then*

(i) *all the projections $p^m : A_m \to \varinjlim A$ are Lipschitz maps,*

(ii) *if* card $A_m > 1$ *for every m, then*

$$\limsup_m \mathrm{Lip}(p^m) \leqslant 1;$$

if, moreover, $\mathrm{Lip}(A) = \prod_{i=1}^\infty \mathrm{Lip}(p_{i+1}^i)$, *then*

$$\lim_m \mathrm{Lip}(p^m) \leqslant 1.$$

Proof. Let $\lambda_i = \mathrm{Lip}(p_{i+1}^i)$ for every i. For any $x, y \in A_m$,

$$\begin{aligned}\rho(p^m(x), p^m(y)) &= \rho\Big(\lim_{n\geqslant m} p_n^m(x), \lim_{n\geqslant m} p_n^m(y)\Big)\\ &= \lim_{n\geqslant m} \rho(p_n^m(x), p_n^m(y)) \leqslant \liminf_{n\geqslant m} \mathrm{Lip}(p_n^m)\rho(x, y)\\ &\leqslant \liminf_{n\geqslant m} \prod_{i=m}^{n-1} \mathrm{Lip}(p_{i+1}^i)\rho(x, y) = \liminf_{n\geqslant m} \prod_{i=m}^{n-1} \lambda_i \rho(x, y).\end{aligned}$$

This proves (i).

If card $A_i > 1$, then $\lambda_i > 0$; since

$$\prod_{i=m}^{n-1} \lambda_i = \left(\prod_{i=1}^{n-1} \lambda_i\right)\left(\prod_{i=1}^{m-1} \lambda_i\right)^{-1},$$

it follows that

$$\mathrm{Lip}(p^m) \leqslant \left(\prod_{i=1}^{m-1} \lambda_i\right)^{-1} \mathrm{Lip}(A).$$

This completes the proof. □

The following result is an analogue of Theorem 2.3.

Theorem 4.3. *Let $A = (A_n, p_{n+1}^n)$ be a Lipschitz direct sequence in (X, ρ), with all A_n being continua and all the threads equiconvergent. Let $s > 0$ and $\mathscr{H}^s(A_n) < \infty$ for every n. Then*

$$\limsup_n \mathscr{H}^s(A_n) \geqslant \mathscr{H}^s(\varinjlim A).$$

Proof. Let $A = \varinjlim A$. By Proposition 4.2(i), all $p^n : A_n \to A$ are Lipschitz maps.

Let $\alpha_n = \mathrm{Lip}(p^n)$ for every n. By [1, 1.8],

$$\mathscr{H}^s(A) \leqslant (\alpha_n)^s \mathscr{H}^s(A_n),$$

whence, by Proposition 4.2(ii),

$$\mathscr{H}^s(A) \leqslant \limsup_n \mathscr{H}^s(A_n). \qquad \square$$

5. Some sequences of sets with positive reach

Federer in [2] introduced the notion of reach of a subset of $\mathbb{R}^k$ with Euclidean metric ρ. For compact sets his definition is equivalent to the following.

Let A be a compact nonempty subset of $\mathbb{R}^k$ for some $k \geqslant 1$. For any $\alpha > 0$ let A_α be the (outer) parallel body of A at distance α, that is (as in (3.4), (3.5))

$$A_\alpha := \{x \in \mathbb{R}^k : \rho(x, A) \leqslant \alpha\}.$$

Let, further,

$$U(A) := \{x \in \mathbb{R}^k : \text{there is a unique } a \in A \text{ with } \rho(x, a) = \rho(x, A)\}.$$

Then

$$\mathrm{reach}(A) := \sup\{\alpha : \mathrm{Int}(A_\alpha) \subset U(A)\}.$$

Clearly, $U(A)$ is the domain of the metric projection $\xi_A : U(A) \to A$, which assigns to an x the unique $a \in A$ with $\rho(x, a) = \rho(x, A)$.

In [5, Theorem 4.8 and Corollary 4.9] we were concerned with convex sets. It is well known that a compact set $A \subset \mathbb{R}^k$ is convex if and only if $\mathrm{reach}(A) = \infty$ (see [2, p. 433]). Here we replace convex sets by sets with positive reach. The following statement due to Federer is a counterpart of the Busemann–Feller Lemma (see [4, p. 35] or [5, 4.7]), which states that for a convex $A \subset \mathbb{R}^k$ the metric projection $\xi_A : \mathbb{R}^k \to A$ is a weak contraction.

Proposition 5.1 (see [2, 4.8(8)]). *Let A be a compact subset of $\mathbb{R}^k$ with* $\operatorname{reach}(A) \geqslant \beta$ *and let* $0 < \alpha < \beta$. *Then the restriction* $\xi_A \mid A_\alpha$ *of the metric projection* ξ_A *is Lipschitzian with* $\operatorname{Lip}(\xi_A \mid A_\alpha) \leqslant \beta \cdot (\beta - \alpha)^{-1}$.

The idea of proof of the next theorem is analogous to that of Theorem 4.8 in [5]; instead of the Busemann–Feller Lemma we use the above statement 5.1.

Theorem 5.2. *Let (A_n) be a sequence of compact subsets of $\mathbb{R}^k$ for some $k \geqslant 2$. If for some sequences (α_n) and (β_n) of real numbers*

(i) *there exists $\delta > 0$ such that $0 < \alpha_n \leqslant \beta_n - \delta$ for every n,*

(ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$,

(iii) $A_{n+1} \subset (A_n)_{\alpha_n}$ *and* $\operatorname{reach}(A_n) \geqslant \beta_n$ *for every n,*

then the formula

$$p_n^{n+1}(x) = \xi_{A_n}(x) \quad \text{for } x \in A_{n+1} \tag{5.1}$$

defines maps $p_n^{n+1} : A_{n+1} \to A_n$ such that $A = (A_n, p_n^{n+1})$ is a Lipschitz geometric inverse sequence and

$$\operatorname{Lip}(A) = \prod_{n=1}^{\infty} \operatorname{Lip}(p_n^{n+1}).$$

Proof. By (i) and (iii),

$$(A_n)_{\alpha_n} \subset U(A_n)$$

and we can define p_n^{n+1} by the formula (5.1).

By Proposition 5.1, the map p_n^{n+1} is Lipschitzian with

$$\operatorname{Lip}(p_n^{n+1}) \leqslant \beta_n \cdot (\beta_n - \alpha_n)^{-1}.$$

We shall prove that

$$\prod_{n=1}^{\infty} \operatorname{Lip}(p_n^{n+1}) < \infty. \tag{5.2}$$

Indeed, (i) and (ii) imply

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{\beta_n - \alpha_n} < \infty. \tag{5.3}$$

Since $\alpha_n/(\beta_n - \alpha_n) \geqslant 0$ and $1 + \alpha_n/(\beta_n - \alpha_n) = \beta_n/(\beta_n - \alpha_n)$, condition (5.3) is equivalent to

$$\prod_{n=1}^{\infty} \frac{\beta_n}{\beta_n - \alpha_n} < \infty,$$

which implies (5.2).

Let $A = (A_n, p_n^{n+1})$. It remains to prove that A is geometric. By Theorem 2.2 it suffices to check that

$$\text{all } A_n \text{ are contained in a compact set } X \subset \mathbb{R}^k \tag{5.4}$$

and

the threads of A are equiconvergent. (5.5)

Let $\alpha := \sum_{n=1}^{\infty} \alpha_n$ and $X := (A_1)_\alpha$. Then, evidently, X is compact and, by (iii),

$$A_n \subset (A_1)_\alpha = X \quad \text{for every } n,$$

which proves (5.4). To verify (5.5), it suffices to prove that

$$(\forall \epsilon > 0)(\exists n_0)(\forall (x_n) \in \varprojlim A)(\forall n,\, m \geqslant n_0)[\rho(x_n, x_m) < \epsilon].$$

Let $m < n$; then

$$x_m = p_m^n(x_n) = p_m^{m+1} \cdots p_{n-1}^n(x_n).$$

Since

$$x_{i-1} = p_{i-1}^i(x_i)$$

and, by (5.1),

$$\rho(x_i, p_{i-1}^i(x_i)) = \rho(x_i, A_{i-1}) \quad \text{for } i \geqslant 2,$$

we infer that

$$\rho(x_n, x_m) \leqslant \sum_{i=m+1}^{n} \rho(x_i, x_{i-1}) = \sum_{i=m+1}^{n} \rho(x_i, A_{i-1}).$$

Therefore, by (iii),

$$\rho(x_n, x_m) \leqslant \sum_{i=m+1}^{n} \alpha_{i-1} = \sum_{i=m}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \alpha_i - \sum_{i=1}^{m-1} \alpha_i.$$

By (ii), there exists n_0 such that for $n > m \geqslant n_0$

$$\rho(x_n, x_m) < \epsilon. \qquad \square$$

As was mentioned in the Introduction, we are looking for sequences for which the Hausdorff measure is continuous with respect to $\lim_{\mathrm{H}}$. We should like to make use of Theorems [5, 3.5] and 3.5 to replace $\lim_{\mathrm{H}}$ by the metric limit of a suitable inverse or direct sequence with surjective bonding maps, and apply Theorems 2.3 and 4.3. From this point of view, the class of sequences described by Theorem 5.2 is too large, because the bonding maps p_n^{n+1} in Theorem 5.2 need not be surjective. Indeed, if (A_n) is a descending sequence of concentric discs with suitable (distinct) radii, then evidently p_n^{n+1} are inclusion maps which are not surjective. For this reason we shall restrict the class of compact sets with positive reach.

Let us first note the following fact mentioned in [2, p. 464].

Lemma 5.3. *For every subset A of $\mathbb{R}^k$, if* $\operatorname{reach}(A) > \alpha$, *then $\xi_A \mid A_\alpha$ is a deformation retraction.*

Proof. It is obvious that $\xi_A \mid A_\alpha$ is a retraction of A_α onto A. Let

$$\varphi(x, t) := (1-t)x + t\xi_A(x) \quad \text{for every } (x, t) \in A_\alpha \times [0, 1].$$

Then $\varphi(A_\alpha \times [0, 1]) \subset A_\alpha$, because for every $(x, t) \in A_\alpha \times [0, 1]$

$$\rho(\varphi(x, t), A) \leqslant \rho(\varphi(x, t), \xi_A(x)) = |1-t| \rho(x, \xi_A(x)) \leqslant \alpha.$$

Evidently, $\varphi(x, 0) = x$ and $\varphi(x, 1) = \xi_A(x)$ for every $x \in A_\alpha$. $\square$

We shall need the following well-known lemma.

Lemma 5.4. *Let A and B be compact r-dimensional manifolds and let*

$$G = \begin{cases} Z, & \text{if } A \text{ and } B \text{ are orientable}, \\ Z_2, & \text{otherwise}. \end{cases}$$

If a map $f: A \to B$ induces an epimorphism $f_*: H_r(A; G) \to H_r(B; G)$ of rth homology groups over G, then $f(A) = B$.

Proposition 5.5. *Let A_1 and A_2 be compact manifolds in $\mathbb{R}^k$ with* $\dim A_1 = \dim A_2 = r < k$. *If*

$$\rho_H(A_1, A_2) < \alpha, \qquad \operatorname{reach}(A_i) \geqslant \beta \quad \text{for } i = 1, 2, \quad \text{and} \quad \alpha < \tfrac{1}{2}\beta,$$

then

$$\xi_{A_1}(A_2) = A_1 \quad \text{and} \quad \xi_{A_2}(A_1) = A_2.$$

Proof. By the assumptions,

$$A_2 \subset (A_1)_\alpha \subset (A_2)_{2\alpha} \subset U(A_2).$$

Let $\xi_1 = \xi_{A_2} | (A_1)_\alpha$ and let $i_1: A_1 \to (A_1)_\alpha$ be the inclusion map. We have

$$A_1 \xrightarrow{i_1} (A_1)_\alpha \xrightarrow{\xi_1} A_2.$$

Then, by Lemma 5.3, the map i_1 is a homotopy equivalence, whence

$$(i_1)_*: H_r(A_1; Z_2) \to H_r((A_1)_\alpha; Z_2) \text{ is an isomorphism.} \tag{5.6}$$

Since $\xi_{A_2} | (A_2)_{2\alpha}$ is a retraction, so is ξ_1; hence

$$(\xi_1)_*: H_r((A_1)_\alpha; Z_2) \to H_r(A_2; Z_2) \text{ is an epimorphism.} \tag{5.7}$$

Let $f_1 := \xi_{A_2} | A_1$, i.e., $f_1 = \xi_1 \circ i_1$. Then

$$(f_1)_* = (\xi_1)_* \circ (i_1)_*,$$

whence, by (5.6) and (5.7), $(f_1)_*$ is an epimorphism. Therefore, by Lemma 5.4, the map f_1 is surjective, i.e.,

$$\xi_{A_2}(A_1) = A_2.$$

Since the assumptions are symmetric with respect to the indices 1, 2, by interchanging the indices we obtain the equality

$$\xi_{A_1}(A_2) = A_1. \quad \square$$

We are now ready to prove our main results.

Theorem 5.6. *Let (A_n) be a sequence of compact r-dimensional manifolds in $\mathbb{R}^k$ for some $k \geqslant 2$ and $r \in \{1, \dots, k-1\}$. If for some sequences (α_n) and (β_n) of real numbers*

(i) *there exists $\delta > 0$ such that $0 < \alpha_n \leqslant \frac{1}{2}\beta_n - \delta$ for every n,*

(ii) $\sum_{n=1}^{\infty}\alpha_n < \infty$,

(iii) $\rho_{\mathrm{H}}(A_n, A_{n+1}) < \alpha_n$ *and* $\mathrm{reach}(A_i) \geqslant \beta_n$ *for $i = n, n+1$,*

then there exist surjective maps

$$p_n^{n+1}: A_{n+1} \to A_n \quad \text{and} \quad p_{n+1}^n: A_n \to A_{n+1}$$

such that

(a) *(A_n, p_n^{n+1}) is a Lipschitz geometric inverse sequence with* $\mathrm{Lip}(A_n, p_n^{n+1}) = \prod_{i=1}^{\infty}\mathrm{Lip}(p_i^{i+1})$,

(b) *(A_n, p_{n+1}^n) is a Lipschitz direct sequence with equiconvergent threads and with* $\mathrm{Lip}(A_n, p_{n+1}^n) = \prod_{i=1}^{\infty}\mathrm{Lip}(p_{i+1}^i)$,

(c) $\varprojlim (A_n, p_n^{n+1}) = \lim_{\mathrm{H}} A_n = \varinjlim (A_n, p_{n+1}^n)$.

Proof. The sequences (A_n), (α_n), and (β_n) satisfy the assumptions of Theorem 5.2; hence the formula

$$p_n^{n+1} = \xi_{A_n} \mid A_{n+1}$$

defines maps $p_n^{n+1}: A_{n+1} \to A_n$ such that (A_n, p_n^{n+1}) satisfies (a). By Proposition 5.5 these maps are surjective.

Let now $p_{n+1}^n: A_n \to A_{n+1}$ be defined by the formula

$$p_{n+1}^n = \xi_{A_{n+1}} \mid A_n.$$

Since the assumption (iii) is symmetric with respect to the indices n, $n+1$, the same argument proves that (A_n, p_{n+1}^n) satisfies (b). By Proposition 5.5, each p_{n+1}^n is surjective.

Finally, applying [5, 3.5] and Theorem 3.5, we obtain (c). □

Corollary 5.7. *Let (A_n) be a Hausdorff convergent sequence of compact r-dimensional manifolds in $\mathbb{R}^k$ for some $k \geqslant 2$ and $r \in \{1, \dots, k-1\}$. If there exists $\epsilon > 0$ such that* $\mathrm{reach}(A_n) \geqslant \epsilon$ *for all n, then*

$$\lim_n \mathscr{H}^r(A_n) = \mathscr{H}^r(\lim_{\mathrm{H}} A_n).$$

Proof. Since all A_n have positive reach, they are of class C^1 (see [2]) and thus their Hausdorff dimension is equal to r and $0 < \mathscr{H}^r(A_n) < \infty$ for every n.

Let $A = \lim_{\mathrm{H}} A_n$. Since $\mathrm{reach}(A_n) \geqslant \epsilon$ for every n, in view of Remark 4.20 in [2], A is again a compact r-dimensional manifold with $\mathrm{reach}(A) \geqslant \epsilon$; thus $0 < \mathscr{H}^r(A) < \infty$.

Let (α_n) be a sequence of positive numbers with $\sum_{n=1}^{\infty}\alpha_n < \infty$ and let $0 < \delta < \frac{1}{2}\epsilon$. Let, further,

$$\beta_n = \min\{\mathrm{reach}(A_n), \mathrm{reach}(A_{n+1})\} \quad \text{for every } n.$$

Then $\frac{1}{2}\beta_n > \delta$ and there exists n_0 such that

$$\alpha_n < \tfrac{1}{2}\beta_n - \delta \quad \text{and} \quad \rho_{\mathrm{H}}(A_n, A_{n+1}) < \alpha_n \tag{5.8}$$

for $n \geqslant n_0$.

Evidently, without any loss of generality, we may assume that (5.8) holds for every n. Then (α_n) and (β_n) satisfy conditions (i)–(iii) of Theorem 5.6. By Theorem 5.6 combined with Theorem 2.3,

$$\lim_n \mathscr{H}^r(A_n) \leqslant \mathscr{H}^r(A).$$

On the other hand, by Theorem 5.6 combined with Theorem 4.3,

$$\lim_n \mathscr{H}^r(A_n) \geqslant \mathscr{H}^r(A).$$

Thus the proof is complete. □

6. Final remarks

(A) The notions of direct sequence and direct limit in arbitrary category **K** are dual to the notions of inverse sequence and inverse limit in **K**. There is an evident analogy between metric limit of an inverse sequence in (X, ρ) and metric limit of a direct sequence. In [5] and in Section 2 of the present paper we are concerned in geometric inverse sequences, which are characterized by two properties:

(1) metric limit $\underleftarrow{\underleftarrow{\lim}}\, A$ is canonically homeomorphic to the topological inverse limit $\underleftarrow{\lim}\, A$ and

(2) all the threads of A are equiconvergent.

The first property is important from the point of view of category theory: we want to assure $\underleftarrow{\underleftarrow{\lim}}$ to be an inverse limit in **Top**. The second is important from the point of view of geometry: we want to link the metric limit $\underleftarrow{\underleftarrow{\lim}}$ with the Hausdorff limit $\lim_{\mathrm{H}}$.

Passing to direct limits, we neglect the first property, because we do not see any reasonable examples or applications. While every Lipschitz inverse sequence satisfies (1) (see Theorem 2.2), for direct sequences the corresponding class would consist of "co-Lipschitz" direct sequences whose bonding maps satisfy the condition:

$$(\exists \lambda > 0)(\forall x, y)\left[\rho(p^n_{n+1}(x), p^n_{n+1}(y)) \geqslant \lambda \rho(x, y)\right].$$

We are not interested here in such direct sequences. Thus we concentrate on the second property, (2), and prove that it plays the same role for direct as for inverse sequences (see [6, 0.3.5] and Theorem 3.5 above).

(B) According to Theorem 4.2 of [5], if A is an inverse sequence in the category **Metr**, with all the threads convergent and the quotient maps continuous, then $\underleftarrow{\underleftarrow{\lim}}\, A$ is an inverse limit of A in **Metr**.

Proposition 1.4 of [5] combined with Lemma 1.4(i) yields the following.

Proposition 6.1. *If A is a Lipschitz inverse sequence with all the threads convergent and the quotient map continuous, then $\underleftarrow{\underleftarrow{\lim}}\, A$ is an inverse limit of A in* $\mathbf{Top}_0$.

The natural question arises, whether Theorem 4.2 of [5] has an analogue for the category **Lip**, in particular, whether $\mathbf{Top}_0$ may be replaced by **Lip** in the above Proposition 6.1. The following example gives the negative answer to this particular question. It was sketched by the referee of the first version of this paper; details are due to K. Rudnik.

Example 6.2. Consider the following two sequences (b_n) and (c_n) in $\mathbb{R}$:

$$b_n = 2 - \frac{3}{2^n} \quad \text{and} \quad c_n = 2 - \frac{1}{2^{n-1}} \quad \text{for } n \in \mathbb{N}.$$

We have

$$\rho(b_n, c_n) = \frac{1}{2^n} \quad \text{and} \quad \rho(b_n, c_{n+1}) = \frac{1}{2^{n-1}}. \tag{6.1}$$

Let $A_n = [0, c_n]$ for every n. Then

$$A_{n+1} = [0, b_n] \cup [b_n, c_{n+1}]$$

and (6.2)

$$[b_n, c_n] \subset [b_{n-1}, c_n].$$

Let $h_n : [b_n, c_{n+1}] \to [b_n, c_n]$ be the homothety with centre b_n. Its ratio is $\operatorname{Lip}(h_n) = \frac{1}{2}$. By (6.2) we can define $p_n^{n+1} : A_{n+1} \to A_n$ by the formula

$$p_n^{n+1}(x) = \begin{cases} x, & \text{if } x \in [0, b_n] \\ h_n(x), & \text{if } x \in [b_n, c_{n+1}] \end{cases} \quad \text{for } n \geqslant 2. \tag{6.3}$$

Let $A = (A_n, p_n^{n+1})$. Since h_n is a contraction, it follows that

$$\operatorname{Lip}(p_n^{n+1}) = \operatorname{Lip}(p_n^{n+1} | [0, b_n]) = 1 \quad \text{for every } n.$$

Thus A is a Lipschitz inverse sequence with $\operatorname{Lip}(A) = 1$. Moreover, all $(p_n^{n+1})^{-1}$ are Lipschitzian and

$$\operatorname{Lip}(p_n^{n+1})^{-1} = \operatorname{Lip}(h_n)^{-1} = 2. \tag{6.4}$$

Evidently, the threads of A are convergent and $\varprojlim A = [0, 2]$. Thus, in view of Theorem 2.2, the sequence A is geometric, whence the quotient map is a homeomorphism. Hence $\varprojlim A$ is an inverse limit of A in $\mathbf{Top}_0$. We define a map $f = (f_n) : Y \to A$ such that all f_n are Lipschitzian but $\varprojlim f$ is not: let

$$Y = [0, 1] = A_1 \quad \text{and} \quad f_n = (p_1^n)^{-1};$$

then $f = (f_n)$ is a map of inverse sequences. From (6.4) and (6.3) it can be deduced that

$$\operatorname{Lip}(f_n) = 2^{n-1}. \tag{6.5}$$

Let $f = \varprojlim f$. To prove that f is not Lipschitzian, it suffices to find (x_n) and (y_n) such that

$$\rho(f(x_n), f(y_n)) > 2^{n-1}\rho(x_n, y_n) \quad \text{for every } n. \tag{6.6}$$

Let $x_n = 1$ and $y_n = p_1^n(b_n)$. Then

$$\rho(x_n, y_n) = \rho(p_1^n(c_n), p_1^n(b_n)) = \frac{1}{4^n}$$

and

$$\rho(f(x_n), f(y_n)) = \rho(2, b_n) = \frac{3}{2^n},$$

which proves (6.6).

7. References

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